Aide-mémoire

Métriques: $ds^2 = -d\tau^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$

Minkowski (RR) $(\eta^{\mu\nu}) = (\eta_{\mu\nu}) = (-1, 1, 1, 1)$

coord. cart : $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

coord. sph. : $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$

Notation matricielle: V = vecteur, M = tenseur $(1,1)$

$$(V^\mu)^{\mu}_{\downarrow} = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix}, \quad \underset{\mu\to}{(M^\nu{}_\mu)^{\nu}_{\downarrow}} = \begin{pmatrix} M^0_0 & M^0_1 & M^0_2 & M^0_3 \\ M^1_0 & M^1_1 & M^1_2 & M^1_3 \\ M^2_0 & M^2_1 & M^2_2 & M^2_3 \\ M^3_0 & M^3_1 & M^3_2 & M^3_3 \end{pmatrix}$$

ω = 1-forme: $\underset{\nu\to}{(\omega_\nu)} = (\omega_0, \omega_1, \omega_2, \omega_3)$

alors : $M(\omega, V) = \omega_\nu M^\nu{}_\mu V^\mu = (\omega_\nu)\cdot(M^\nu{}_\mu)\cdot(V^\mu)$

Transf. de Lorentz: $\Delta x^{\alpha'} = \Lambda^{\alpha'}{}_\beta\,\Delta x^\beta, \quad \hat{e}_{(\alpha)} = \Lambda^{\beta'}{}_\alpha \hat{e}_{(\beta')}$

$$(\Lambda^{\alpha'}{}_\beta) = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \left(1 - v^2\right)^{-1/2}$$

4-vitesse, 4-accélération, 4-impulsion, 4-force :

$u^\alpha = \frac{dx^\alpha}{d\tau},\ u\cdot u = -1,\ a^\alpha = \frac{d^2x^\alpha}{d\tau^2} = \frac{du^\alpha}{d\tau},\ u\cdot a = 0$

$p^\mu = m u^\mu = (E, p),\ p\cdot p = m^2 u\cdot u = -m^2 = -E^2 + p^2$

$m \neq 0 :\ E = \gamma m, \quad p = \gamma m v, \qquad E_{\rm obs} = -p\cdot u_{\rm obs}$

$m = 0 :\ E = |\mathbf{p}| = h\nu$ (photon)

$f^\mu = m\frac{d^2}{d\tau^2}x^\mu(\tau) = \frac{d}{d\tau}p^\mu(\tau)$

Tenseur énergie-impulsion et flux :

$T^{\mu\nu}_{\text{poussière}} = p^\mu N^\nu = (\rho + p)u^\mu u^\nu + p\eta^{\mu\nu}$.

où $T^{\alpha\beta} = T^{\beta\alpha}, \quad T^{\alpha\beta}_{;\beta} = 0, \quad N^\alpha_{;\alpha} = (n\,u^\alpha)_{;\alpha} = 0$

Vecteurs et formes : Soit ω = 1-forme et A = vecteur :

$\omega(A) \equiv \langle \omega\,, A\rangle = \omega(A^\alpha \hat{e}_{(\alpha)}) = A^\alpha \omega(\hat{e}_{(\alpha)}) = A^\alpha \omega_\alpha$

$\omega_{\beta'} = \Lambda^\alpha{}_{\beta'}\,\omega_\alpha, \quad A^{\alpha'} = \Lambda^{\alpha'}{}_\beta\, A^\beta, \quad \hat{e}_{(\alpha)} = \Lambda^{\beta'}{}_\alpha \hat{e}_{(\beta')}$

$\hat{\theta}^{(\alpha)}(\hat{e}_{(\beta)}) = \delta^\alpha{}_\beta \qquad \hat{\theta}^{(\alpha')} = \Lambda^{\alpha'}{}_\beta\ \hat{\theta}^{(\beta)}$

$\frac{d\phi}{d\tau} = \phi_{,\alpha}u^\alpha, \quad \phi_{,\alpha} \equiv \frac{\partial\phi}{\partial x^\alpha}, \quad x_{,\alpha} \equiv \frac{\partial x^\alpha}{\partial x^\beta} = \delta^\alpha_\beta, \quad d\phi = \phi_{,\alpha}dx^\alpha$

$f = f_{\alpha\beta}\,\hat{\theta}^{(\alpha)} \otimes \hat{\theta}^{(\beta)}, \quad f(\hat{\theta}^{(\alpha)}, \hat{\theta}^{(\beta)}) \equiv f^{\alpha\beta}, \quad f(\hat{e}_{(\alpha)}, \hat{e}_{(\beta)}) \equiv f_{\alpha\beta}$

$q^\alpha = \langle \hat{\theta}^{(\alpha)}, q\rangle, \quad \sigma_\alpha = \langle \sigma, \hat{e}_{(\alpha)}\rangle$

$b_{\alpha\beta} = \frac{1}{2}\left(b_{\alpha\beta} + b_{\beta\alpha}\right) + \frac{1}{2}\left(b_{\alpha\beta} - b_{\beta\alpha}\right) = b_{(\alpha\beta)} + b_{[\alpha\beta]}$

$S = S^\beta_{\alpha\gamma}\,\hat{\theta}^{(\alpha)} \otimes \hat{e}_{(\beta)} \otimes \hat{\theta}^{(\gamma)}, \quad S(u, \sigma, v) = S^\beta_{\alpha\gamma}\, u^\alpha \sigma_\beta v^\gamma$

$Q^{\alpha'}_{\beta'} = Q(\hat{\theta}^{(\alpha')}, \hat{e}_{(\beta')}) = Q(\Lambda^{\alpha'}{}_\beta\ \hat{\theta}^{(\beta)}, \Lambda^\nu{}_{\beta'}\ \hat{e}_{(\nu)}) = \Lambda^{\alpha'}{}_\beta\ \Lambda^\nu{}_{\beta'}\ Q^\beta_\nu$

$V_\alpha = g_{\alpha\beta}\,V^\beta, \quad V^\alpha = g^{\alpha\beta}\,V_\beta, \quad \hat{e}_{(\alpha)} = \frac{\partial}{\partial x^\alpha}, \quad \hat{\theta}^{(\alpha)} = \mathrm{d}x^\alpha$

Base orthonormée naturelle et tétrades: $g_{\mu\nu}e^\mu_a e^\nu_b = \eta_{ab},\ g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$

$\hat{e}_{(\mu)} = e^a_\mu \hat{e}_{(a)},\ \hat{\theta}^{(\mu)} = e^\mu_a \hat{\theta}^{(a)}, \quad e^\mu_a e^a_\nu = \delta^\mu_\nu, \quad e^a_\mu e^\mu_b = \delta^a_b.$

$2D$ polaires: $\hat{\theta}^{(\hat{r})} = dr,\ \hat{\theta}^{(\hat{\theta})} = r\,d\theta,\ \hat{e}_{(\hat{r})} = \hat{e}_{(r)},\ \hat{e}_{(\hat{\theta})} = r^{-1}\hat{e}_{(\theta)}$

Dérivée covariante et symbole de Christoffel:

$\nabla_\beta V^\alpha \equiv V^\alpha_{\ ;\beta} \equiv V^\alpha_{\ ,\beta} + \Gamma^\alpha{}_{\mu\beta}\,V^\mu, \quad p_{\alpha;\beta} = p_{\alpha,\beta} - \Gamma^\mu{}_{\alpha\beta}\,p_\mu,$

$T^\alpha{}_{\beta;\gamma} = T^\alpha{}_{\beta,\gamma} + \Gamma^\alpha{}_{\mu\gamma}T^\mu{}_\beta - \Gamma^\mu{}_{\beta\gamma}T^\alpha{}_\mu$

$\frac{\partial\hat{e}_{(\alpha)}}{\partial x^\beta} \equiv \Gamma^\mu{}_{\alpha\beta}\,\hat{e}_{(\mu)}, \quad \frac{D}{d\lambda}V^\alpha = \frac{dx^\beta}{d\lambda}\nabla_\beta V^\alpha = \frac{dV^\alpha}{d\lambda} = V^\alpha_{\ ;\beta}u^\beta$

$\Gamma^\alpha{}_{\gamma\beta} = \Gamma^\alpha{}_{\beta\gamma}, \qquad g_{\alpha\beta;\gamma} = 0,$

$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\,g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right)$

$d\,x^0\,d\,x^1\,d\,x^2\,d\,x^3 = (-g)^{1/2}\,d\,x^{0'}\,d\,x^{1'}\,d\,x^{2'}\,d\,x^{3'}$

Géodésique: ($\lambda = \tau$ pour type temps)

$\frac{D}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0,\ u^\alpha = \frac{dx^\alpha}{d\lambda} \equiv \dot{x}^\alpha,\ \frac{d}{d\lambda} = u^\beta\frac{\partial}{\partial x^\beta}$

$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0, \quad u^\mu\nabla_\mu u^\nu = 0, \quad p^\mu\nabla_\mu p^\nu = 0$

$p^\alpha\,p^\beta_{\ ;\alpha} = 0 \Rightarrow p^\alpha\,p_{\beta;\alpha} = 0 \Rightarrow m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}\,p^\nu\,p^\alpha$

Tenseur de Riemann, de Ricci et scalaire de Ricci

$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\sigma\mu}\Gamma^\sigma{}_{\beta\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\beta\mu}$

$R_{\alpha\beta} = R^\mu{}_{\alpha\mu\beta}, \qquad R = R^\alpha{}_\alpha \qquad g_{\mu\nu} = g_{\nu\mu}$

$R_{\mu\nu} = R_{\nu\mu},\ R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$

Tenseur et éq. d'Einstein

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}, \quad G^{\alpha\beta}{}_{;\beta} = 0, \quad G^{\alpha\beta} = 8\pi\,T^{\alpha\beta}$

Schwarzschild (Sch) : avec $\Delta = 1 - \frac{2GM}{r}$

$ds^2\big|_{\text{Sch}} = -\Delta \mathrm{d}t^2 + \Delta^{-1}\,\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)$

Géodésiques: Soit $V(r) = \frac{1}{2}\epsilon - \epsilon\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3}$

$\frac{dt}{d\lambda} = \Delta^{-1}\,E\ , \quad \frac{d\phi}{d\lambda} = \frac{L}{r^2}\ , \quad \frac{dr}{d\lambda} = \pm\left[E^2 - 2V(r)\right]^{1/2}$

Particules $m \neq 0$: $\lambda = \tau$, $\epsilon = 1$ = type temps

$\frac{dt}{d\tau} = \frac{p^t}{m} = \frac{g^{tt}p_t}{m} = \gamma\ , \quad \frac{d\phi}{d\tau} = \frac{p^\phi}{m} = \frac{g^{\phi\phi}p_\phi}{m}\ , \quad \frac{dr}{d\tau} = \frac{p^r}{m}$

Photons ou $m = 0$: $\epsilon = 0$ = type nul

$\frac{dt}{d\lambda} = p^t = g^{tt}p_t\ , \quad \frac{d\phi}{d\lambda} = p^\phi = g^{\phi\phi}p_\phi\ ,$

$\frac{dr}{d\lambda} = p^r = \frac{\nu}{\nu_\infty} = \frac{\lambda_\infty}{\lambda} = \Delta^{-1/2}$

Reissner-Nordstrøm (RN) :

$ds^2\big|_{\text{RN}} = ds^2\big|_{\text{Sch}}$ avec $\Delta = 1 - \frac{2GM}{r} + \frac{G(p^2+q^2)}{r^2}$

Kerr :

$ds^2\big|_{\text{Kerr}} = -\mathrm{d}t^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\phi^2 + \frac{2GMr}{\rho^2}(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t)^2$

avec $\Delta = r^2 - 2GMr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$ et $a = J/M$